ANNOTATION
On the dissertation for the degree of doctor of Philosophy (PhD) in specialty «6D060100-Mathematics»
Zhanuzakova Dinara Taupikhovna
«Methods of approximate solution of direct and inverse problems of filtration theory»

The relevance of the research topic. This dissertation is devoted to the study of direct and inverse problems for some mathematical models arising in the filtration theory, that is, the mathematical model of the filtration theory that takes into account the phase transition, the initial boundary value problem for the pseudoparabolic equation, the inverse problem for the parabolic equation, and the reaction-diffusion problem are considered. Equations of this type and general equations of Sobolev type appear in the description of heat and mass transfer processes in filtration theory, moisture transport in soil, plasma physics, population and many other fields.

The aim of the PhD thesis. Study of direct and inverse problems of the filtration theory. In particular:

- The correctness of the mathematical model of the Stefan-type problem, the study of the existence and uniqueness of the solution, the transition to the limit in terms of relaxation time.

- Study of the unique solvability of the initial boundary value problems for the quasilinear pseudoparabolic equation given by the nonlinear boundary condition. Proving the theorem on the existence and uniqueness of a weakly generalized solution of the problem, proving the blow-up of the solution of the problem for a quasilinear pseudoparabolic equation with a nonlinear boundary condition, and studying the asymptotic stability of the solution in time.

- Finding a weak solution of the inverse problem for the quasilinear parabolic equation, studying the stability of the solution, and also proving that the solution blow-up in finite time.

- Proving the theorem for finding solutions of a competitive system with nonlinear cross-diffusion for one-phase and two-phase patterns. Finding specific areas of solutions.

Scientific novelty of the work. All the results obtained in the thesis are new and rigorously proven. The obtained results are of theoretical and practical importance and can be used in creating a general theory of direct and inverse problems.

- Existence and uniqueness of a solution of one model of Stefan-type filtration theory with phase relaxation was obtained. The lemma of transition to the limit in terms of relaxation time is proved. Numerical experiments are presented.

- The theorem on the existence and uniqueness of the generalized weak solution of the initial-boundary value problem for the quasilinear equation of the pseudoparabolic type with the Neumann-Dirichlet type nonlinear boundary condition is proved. At the same time, sufficient conditions were obtained for the final destruction of their solutions.

- The existence of a weak solution of the inverse problem of the quasi-linear parabolic type equation with the integral condition given by redefinition was proved by the Galerkin method and the stability of the solution was obtained.

- It was shown that there are two different types of periodic stationary solutions for some range of parameters of the nonlinear cross-diffusion competitive system. The eight-dimensional parameter space is partitioned and the Turing regions of the solutions are shown. In addition to analytical methods, numerical methods for studying the properties of solutions were proposed.

The methods of scientific research. Galerkin's approximation method, a priori estimation method, Sobolev space theory, integral and differential calculus methods, functional analysis method, compactness method, monotonicity method, necessary interpolation inequalities, Jung's, Gelder's and Minkowski's inequalities were used to achieve the research results.

Theoretical and practical significance of the results. The obtained results are of theoretical and practical importance, and the obtained results are primarily of theoretical interest. For the achieved results of the research, it is possible to conduct computational experiments and obtain numerical values of solutions.

Publications. Based on the results of the dissertation research, 12 works were published, including:

– 3 articles in scientific journals included in the fourth and first quartiles (Q4 and Q1) of the Clarivate Analytics Journal Citation Reports, respectively, and/or with CiteScore percentiles of 35, 91, 96 [97, 98] in the Scopus database, respectively:

1. Aitzhanov S.E., Zhanuzakova D.T. Behavior of solutions to an inverse problem for a quasilinear parabolic equation // Siberian Electronic Mathematical Reports. -2019. –Vol. 16. –P.1366-1382. DOI 10.33048/SEMI.2019.16.097 (Scopus: the percentile – 35%, Web of Science: Q4, SJR – 0.516).

2. Aitzhanov S.E., Zhanuzakova D.T. An initial boundary value problem for a pseudoparabolic equation with a nonlinear boundary condition // Mathematical Methods in the Applied Sciences. 2022. DOI 10.1002/mma.8568 (Scopus: the percentile – 91%, Web of Science: Q1, SJR – 0.702).

3. Kersner R., Klincsik M., Zhanuzakova D.T.   A competition system with nonlinear cross-diffusion: exact periodic patterns//Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales-Serie A: Mathematicas. 2022. DOI 10.1007/s13398-022-01299-1 (Scopus: the percentile – 96%, Web of Science: Q1, SJR – 1.055).

– 3 articles in journals recommended by the quality assurance committee in the field of education and science of the Ministry of Education and Science of the Republic of Kazakhstan.

1. Айтжанов С.Е., Жанузакова Д.Т. Разрушение решений обратной задачи для уравнения теплопроводности со степенной нелинейностью // Хабаршы, KazNPU named after Abai, 63, #3, 2018.

2. Айтжанов С.Е., Жанузакова Д.Т. Разрушение решений обратной задачи для параболического уравнения//ҚазҰТУ хабаршысы; Technical Sciences Series, 3, #133, 2019.

4. Мухамбетжанов С.Т., Жанузакова Д.Т. О корректности одной модели теории фильтрации типа Стефана//Хабаршы, KazNPU named after Abai, 65#1, 2019

– 7 publications in the collection of international conferences:

1. Обоснование метода фиктивных областей для модели Маскета-Леверетта// International Scientific and Methodological Conference "Mathematics in Kazakhstan - Past and Prospects", devoted to the 100th anniversary of Ibrashev Hasan Ibrashevich, 2016

2. On the Application of Quasi-Conformal Mappings to Solve the Problem of Filtration Theory//The 5th Abu Dhabi University Annual International Conference Mathematical Science and its Applications, 2017

3. About One Problem of the Isothermal Filtration Process//The 6th Abu Dhabi University Annual International Conference Mathematical Science and its Applications, 2017

4. On the Development of a Mathematical Model of Nonequilibrium Filtration //The 7th Abu Dhabi University Annual International Conference Mathematical Science and Its Applications, 9-12 May 2018

5. Разрушение решений обратной задачи для параболического уравнения со степенной нелинейностью// Actual problems of analysis, differential equations and algebra, 2019

6. Correctness of a one mathematical model of nonequlibrium phase transitions of water in porous media// Rroblems of Differential equations, Analysis and Algebra VIII International Scientific Conference, 1 November , 2018

7. An initial boundary value problem for a pseudoparabolic equation with a nonlinear boundary condition// "Inverse and ill-posed problems in natural sciences" Material of the international scientific conference, 11-12 april, 2023

The structure and scope of the thesis. The dissertation consists of an introduction, 5 parts (each part is divided into sub-parts), conclusion and a list of used literature. There are 8 pictures in the work. The volume of the thesis is 105 pages.

The main content of the thesis. In the introduction of the proposed dissertation, the relevance and novelty of the work is presented, the purpose of the work is formulated, the main rules are established, and a summary of the dissertation is also given.

In the first section, necessary notations are introduced, necessary definitions, well-known lemmas, theorems, and basic inequalities are given.

In the second section, the mathematical model of the theory of filtration, which takes into account the phase transition, is considered:

$$\frac{\partial s}{\partial t} = \frac{1}{\tau}(H(c) - s),$$
$$m \cdot \frac{\partial c}{\partial t} = D \cdot \Delta c - v \cdot \nabla c - \frac{\partial s}{\partial t}.$$

It is intended to find the functions $c(x, t), s(x, t)$ (concentration of surfactant in liquid and solid phases) that satisfy this mathematical model with phase

relaxation. The plausibility of this model has been verified. The existence and uniqueness of the solution are proved. Numerical experiments are presented..

The third section is devoted to the study of the qualitative property of solving the problem of carbonated liquid. The fundamental problem of studying the solution of an initial-boundary value problem for a pseudoparabolic equation in a domain with a sufficiently smooth boundary is considered:

$$\frac{\partial}{\partial t}(u - \chi\Delta u) - \left(a_0 + a_1\|\nabla u\|_{2,\Omega}^{2q-2}\right)\Delta u = b(x,t)|u|^{p-2}u + f(x,t) \ (x,t) \in Q_T,$$

$$\frac{\partial u}{\partial n} + k(x,t)|u|^{\sigma-2}u|_\Gamma = 0, \qquad \Gamma = \partial\Omega \times (0,T),$$

$$u(x,0) = u_0(x), x \in \Omega.$$

In this section, the existence of a weak solution of the initial boundary value problem for the pseudoparabolic equation in the bounded region is proved by the Galerkin method. A priori estimates of the solution were obtained using Sobolev's embedding theorems. On the basis of a priori estimates, the local theorem on the existence and uniqueness of the generalized weak solution of the initial-boundary value problem for the quasi-linear pseudoparabolic equation is proved. Sufficient conditions for the blow-up of the solutions of the Neumann-Dirichlet problem given by the nonlinear boundary condition in the bounded region are obtained.

In the fourth section, the inverse problem imposed on the non-linear parabolic type equation with respect to the degree with the integral condition given by redefinition on the cylinder $Q_T = \{(x,t) : x \in \Omega, \ t \in (0,T)\}$ is considered:

$$\frac{\partial}{\partial t}\left(u + a_0|u|^{p-2}u\right) - \Delta u + a(x,t,u,\nabla u)) = |u|^{p-2}u + f(t)w(x),$$
$$x \in \Omega, \ 0 < t < T$$

$$u(x,0) = u_0(x),$$

$$u|_{\partial\Omega \times (0,T)} = 0,$$

$$\int_\Omega (u + a_0|u|^{p-2}u)\omega \, dx = \varphi(t), \qquad 0 < t < T.$$

It is considered to determine a pair of functions $(u(x,t), f(t))$ satisfying the second-order quasilinear parabolic type inverse problem. The existence of a weak solution of the inverse problem is proved by the Galerkin method. Sufficient conditions for the finite-time decay of the solution in the measured region with a homogeneous Dirichlet condition were obtained, and also the stability of the solution of the nonlinear opposite-sign inverse problem with respect to the power type was obtained.

The fifth section deals with a competitive system with nonlinear cross-diffusion:

$$\begin{cases} u_t = (uu_x + \varepsilon_1 uv_x + \varepsilon_3 vu_x)_x + u(1 - u - cv) := -\dfrac{\partial}{\partial x} J_1 + u(1 - u - cv), \\[2mm] v_t = (dvv_x + \varepsilon_4 uv_x + \varepsilon_2 vu_x)_x + v(a - bu - v) := -\dfrac{\partial}{\partial x} J_2 + v(a - bu - v). \end{cases}$$

Two different reaction-diffusion (RD) systems are studied. It is shown that for some range of parameters there are two different types of periodic stationary solutions. The eight-dimensional parameter space is partitioned by Turing regions with solutions. Some numerical calculations are shown that make it reasonable that the solutions are attractors with a large region of attraction in the space of initial functions.